JOURNAL OF MATHEMATICAL ANALYSIS AND APPLICATIONS **197**, 14–22 (1996)
ARTICLE NO. 0002

A New Characteristic of Möbius Transformations by Use of Apollonius Points of Triangles

Hiroshi Haruki

Department of Pure Mathematics, University of Waterloo, Waterloo, Ontario N2L 3G1, Canada

Themistocles M. Rassias

Department of Mathematics, University of La Verne, P.O. Box 51105, Kifissia, Athens, Greece 14510

Submitted by William F. Ames

Received April 21, 1994

DEDICATED TO PROFESSOR J. ACZÉL ON THE OCCASION OF HIS 70TH BIRTHDAY

The purpose of this paper is to give a new characterization of Möbius transformations from the standpoint of conformal mapping. To this end a new concept of "Apollonius points of triangles" is applied. Throughout the paper, unless otherwise stated, let $w = f(z)$ be a nonconstant meromorphic function of a complex variable z in $|z| < +\infty$.

1. INTRODUCTION: APOLLONIUS POINTS OF TRIANGLES AND THE PRINCIPLE OF CIRCLE TRANSFORMATION

LEMMA 1 (The Theorem of Apollonius). *The locus of a point P moving in a plane such that the ratio of its distance from point A to its distance from point B of the plane is a positive constant k is as follows*:

(i) *In the case when $k \neq 1$ the locus is the circle on IE as diameter, where I and E divide the segment AB internally and externally in the ratio k.*

(ii) *In the case when $k = 1$ the locus is the perpendicular bisector of the segment AB.*

Proof. See [6, pp. 110–111]. ∎

0022-247X/96 $12.00

Remark 1. The circle of Lemma 1 is said to be the circle of Apollonius of points A and B for the ratio k (including straight lines among circles).

In the following the symbol $\overline{BC}$ means the length of the segment BC.

DEFINITION 1. Let ΔABC be an arbitrary triangle and L a point on the complex plane. We denote by $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. If $ax = by = cz$ holds, then L is said to be an Apollonius point of ΔABC. The following theorem holds:

THEOREM 1. *Let ΔABC be an arbitrary triangle on the complex plane. Then the number of Apollonius points of ΔABC is at most* 2.

Proof. The property follows from the Theorem of Apollonius (see Lemma 1) and from the fact that if two circles meet, including straight lines among circles, then there are at most two points of intersection. ▌

EXAMPLE 1. Let ΔABC be an arbitrary equilateral triangle. Then its centroid is its only Apollonius point.

Proof. We denote by $a = \overline{BC} = \overline{CA} = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$, where L is an Apollonius point of ΔABC.

By the definition of Apollonius points of triangles we have

$$ax = ay = az$$

and so

$$x = y = z.$$

Hence L is the centroid of ΔABC. Q.E.D.

EXAMPLE 2. If an isosceles triangle ABC, with equal angles 30° at the ends of its base BC whose midpoint is L_1, is drawn inside an equilateral triangle L_2BC, then L_1, L_2 are the Apollonius points of ΔABC. One of these two points is on a side of ΔABC and the other outside ΔABC.

Proof. First, we prove that L_1 is an Apollonius point of ΔABC. We denote by $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$, $x = \overline{AL_1}$, $y = \overline{BL_1}$, $z = \overline{CL_1}$. If we express b, c, x, y, z in terms of a, then we obtain

$$b = c = \frac{a}{\sqrt{3}}, \qquad x = \frac{a}{2\sqrt{3}}, \qquad y = z = \frac{a}{2}.$$

Hence we have

$$ax = by = cz\left(= \frac{a^2}{2\sqrt{3}} \right).$$

Consequently, by definition L_1 is an Apollonius point of ΔABC.

Similarly we can prove that L_2 is also an Apollonius point of ΔABC.

Q.E.D.

Remark 2. As the above example shows, a point on a side of a triangle may possibly be an Apollonius point of the triangle. However, any vertex of a triangle cannot be one of its Apollonius points.

Before we state Lemma 2 we give the definition of pedal triangles (see [3, p. 22]). Let P be any point on the complex plane, and let perpendiculars PA_1, PB_1, PC_1 be dropped to the three sides BC, CA, AB of ΔABC on the complex plane. The feet of these perpendiculars are the vertices of a triangle $A_1B_1C_1$, which is called the *pedal triangle* of ΔABC for the pedal point P, if A_1, B_1, C_1 are different and not collinear. We may now state Lemma 2.

LEMMA 2. *If the pedal point is distant x, y, z from the vertices A, B, C of ΔABC, respectively, the pedal triangle has sides of lengths*

$$\frac{ax}{2R}, \qquad \frac{by}{2R}, \qquad \frac{cz}{2R},$$

where R denotes the radius of the circumcircle of ΔABC and $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$.

Proof. See [3, p. 23]. ∎

The reader might wish to know the motive of the concept of Apollonius points of triangles. An answer can be obtained from the following theorem.

THEOREM 2. *Let ΔABC be a triangle and L a point on the complex plane. Then L is an Apollonius point of ΔABC iff the pedal triangle of ΔABC for the pedal point L is an equilateral triangle.*

Proof of the If Part. We denote by $a = \overline{BC}$, $b = \overline{CA}$, $c = \overline{AB}$, $x = \overline{AL}$, $y = \overline{BL}$, $z = \overline{CL}$. By hypothesis and Lemma 2 we obtain

$$\frac{ax}{2R} = \frac{by}{2R} = \frac{cz}{2R},$$

where we denote the radius of the circumcircle of ΔABC by R.

Therefore, we obtain

$$ax = by = cz.$$

Hence, by definition L is an Apollonius point of ΔABC, giving the desired result.

Proof of the Only If Part. The proof is similar to that of the if part. ∎

Now, by using Theorem 2 we give two more examples of Apollonius points.

EXAMPLE 3. Let an isosceles triangle ABC, with equal angles $15°$ at the ends of its base BC, be drawn inside an isosceles right-angled triangle L_1BC with hypotenuse BC. Furthermore, we denote the image of A by reflection in the side BC by L_2. Then L_1, L_2 are the Apollonius points of ΔABC. Both are outside ΔABC.

Proof. The proof follows from considering the pedal triangles for the pedal points L_1, L_2 and using Theorem 2. ∎

EXAMPLE 4. Suppose that ΔABC is a triangle on the complex plane such that $< A = 90°$, $< B = 60°$, $< C = 30°$. Let L_1 be a point inside ΔABC satisfying $< BL_1C = 150°$, $< CL_1A = 120°$, $< AL_1B = 90°$ and let L_2 be the image of C by reflection in the side AB. Then L_1, L_2 are the Apollonius points of ΔABC. One of these points is inside ΔABC and the other outside ΔABC.

Proof. The proof is clear from considering the pedal triangles for the pedal points L_1, L_2 and using Theorem 2. ∎

We consider the following Property A:

Property A. $w = f(z)$ transforms circles on the z-plane onto circles on the w-plane, including straight lines among circles.

The well-known principle of circle transformation (see [1], [4], [7, p. 160]) reads as follows:

THEOREM A. *$w = f(z)$ satisfies Property* A *iff $w = f(z)$ is a Möbius transformation.*

LEMMA 3. *If $w = f(z)$ satisfies Property* A, *then $w = f(z)$ is univalent in* $|z| < +\infty$.

Proof. See [4]. The proof does not use Theorem A to avoid that the proof of Theorem A in this paper does not become to be cyclic, but uses the reflection principle for analytic functions. ∎

Before stating Lemma 4 we give the definition of Apollonius quadrilaterals on the complex plane.

DEFINITION 2. Let $ABCD$ be an arbitrary quadrilateral (not necessarily simple) on the complex plane. If $\overline{AB} \cdot \overline{CD} = \overline{BC} \cdot \overline{DA}$ holds, then $ABCD$ is said to be *an Apollonius quadrilateral.*

We may now state Lemma 4.

LEMMA 4. *Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R on the z-plane. Let $ABCD$ be an arbitrary Apollonius quadrilateral contained in R and let $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $D' = f(D)$. If f satisfies Property* A, *then $A'B'C'D'$ is also an Apollonius quadrilateral.*

Proof. See [5]. (The principle of reflection for analytic functions is used.) ∎

LEMMA 5. *If $w = f(z)$ is analytic and univalent in a nonempty domain R, then $f'(z) \neq 0$ in R.*

Proof. See [9. p. 302]. ∎

2. PROOF THAT PROPERTY A IMPLIES PROPERTY B

We state first Property B.

Property B. Suppose that $w = f(z)$ is analytic and univalent in a nonempty domain R of the z-plane. Let ΔABC be an arbitrary triangle contained in R and let its Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, $L' = f(L)$ and if the three different points A', B', C' form a triangle (i.e., A', B', C' are not collinear), then the point L' is also an Apollonius point of $\Delta A'B'C'$.

We may now give a proof that Property A implies Property B.

Let $w = f(z)$ satisfy Property A. Suppose that $w = f(z)$ is analytic in a nonempty domain R on the z-plane. Since, by hypothesis, $w = f(z)$ satisfies Property A, by Lemma 3 $w = f(z)$ is univalent in $|z| < +\infty$. Thus $w = f(z)$ is univalent in R. Let ΔABC be an arbitrary triangle contained in R and let its Apollonius point L be a point of R. If we set $A' = f(A)$, $B' = f(B)$, $C' = f(C)$, then, by the univalency of $w = f(z)$, the three points A', B', C' are different.

We now prove that if A', B', C' are not collinear, then the point $L' = f(L)$ is also an Apollonius point of $\Delta A'B'C'$. Since L is an Apollonius point of ΔABC, by definition we obtain

$$\overline{BC} \cdot \overline{AL} = \overline{CA} \cdot \overline{BL}.$$

Therefore $ABCL$ is an Apollonius quadrilateral. Furthermore, $w = f(z)$ satisfies Property A. Consequently, by Lemma 4 $A'B'C'L'$ is also an Apollonius quadrilateral. Hence we have

$$\overline{B'C'} \cdot \overline{A'L'} = \overline{C'A'} \cdot \overline{B'L'}. \tag{1}$$

Similarly, we have

$$\overline{C'A'} \cdot \overline{B'L'} = \overline{A'B'} \cdot \overline{C'L'}. \tag{2}$$

By (1), (2) we obtain

$$\overline{B'C'} \cdot \overline{A'L'} = \overline{C'A'} \cdot \overline{B'L'} = \overline{A'B'} \cdot \overline{C'L'},$$

which implies by definition that $L' = f(L)$ is also an Apollonius point of $\Delta A'B'C'$. Consequently, $w = f(z)$ satisfies Property B. Q.E.D.

3. PROOF OF THE MAIN THEOREM

The purpose of this article is to give a new invariant characteristic property of Möbius transformations from the standpoint of conformal mapping and to give a new proof of the only if part of Theorem A. The results in this paper are contained in the following theorem and its corollary.

MAIN THEOREM. *$w = f(z)$ satisfies Property* B *iff $w = f(z)$ is a Möbius transformation.*

COROLLARY. *This theorem gives a new proof of the only if part of Theorem A.*

We may now give a proof of the main theorem.

If $w = f(z)$ is a Möbius transformation, then, by the if part of Theorem A, $w = f(z)$ satisfies Property A. Thus, by the result in Section 2, $w = f(z)$ satisfies Property B. Next, we prove the only if part of the main theorem, i.e., if $w = f(z)$ satisfies Property B, then $w = f(z)$ is a Möbius transformation.

Since $w = f(z)$ is analytic and univalent in the domain R, by Lemma 5

$$f'(z) \neq 0 \tag{3}$$

holds in R.

If x is an arbitrarily fixed point of R, then, by (3) we obtain

$$f'(x) \neq 0. \tag{4}$$

Let L be the point represented by x. Since $L \in R$, there exists a positive real number r such that the r closed circular neighborhood of L is contained in R. We denote this closed circular neighborhood by V.

Throughout the proof let ΔABC denote an arbitrary equilateral triangle which is contained in V and whose center is at L. Here the sense of A, B, C is counterclockwise. Since ΔABC is an equilateral triangle contained in V, we can represent A, B, C by complex numbers

$$x + y, x + \omega y, x + \omega^2 y,$$

respectively, where $\omega = (-1 + \sqrt{3}\,\mathrm{i})/2$ and $|y| \leq r$.

Since $w = f(z)$ is univalent in R, $A'(= f(A))$, $B'(= f(B))$, $C'(= f(C))$ are different points. By a property of analytic functions (see [8, p. 16]) and by (4) (A, B, C are not collinear on the z-plane) there exists some sufficiently small positive real number s satisfying

$$s \leq r$$

such that A', B', C' are not collinear on the w-plane for all y satisfying $0 < |y| \leq s$.

Since L is the Apollonius point of the equilateral triangle ΔABC ($0 < |y| \leq s$) (see Example 1 in Section 1) and the three different points A', B', C' are not collinear, by hypothesis $L' = f(L)$ is an Apollonius point of ΔABC. Hence, by definition we obtain

$$\overline{B'C'} \cdot \overline{A'L'} = \overline{C'A'} \cdot \overline{B'L'}. \tag{5}$$

Since A', B', C', L' are represented by

$$f(x+y), f(x+\omega y), f(x+\omega^2 y), f(x),$$

respectively, we obtain

$$\overline{B'C'} = |f(x+\omega y) - f(x+\omega^2 y)|, \tag{6}$$

$$\overline{A'L'} = |f(x+y) - f(x)|, \tag{7}$$

$$\overline{C'A'} = |f(x+\omega^2 y) - f(x+y)|, \tag{8}$$

$$\overline{B'L'} = |f(x+\omega y) - f(x)|. \tag{9}$$

Substituting (6), (7), (8), (9) into (5), we obtain

$$\begin{aligned} &|(f(x+\omega y) - f(x+\omega^2 y))(f(x+y) - f(x))| \\ &\qquad = |(f(x+\omega^2 y) - f(x+y))(f(x+\omega y) - f(x))|, \end{aligned}$$

and so

$$\left| \frac{(f(x+\omega y) - f(x+\omega^2 y))(f(x+y) - f(x))}{(f(x+\omega^2 y) - f(x+y))(f(x+\omega y) - f(x))} \right| = 1. \tag{10}$$

If we set

$$g(y) = \frac{(f(x+\omega y) - f(x+\omega^2 y))(f(x+y) - f(x))}{(f(x+\omega^2 y) - f(x+y))(f(x+\omega y) - f(x))}, \tag{11}$$

then, by (10) we have

$$|g(y)| = 1 \tag{12}$$

in $0 < |y| \leq s$.

Since the numerator and the denominator of $g(y)$ in (11) are analytic functions for all y satisfying $0 < |y| \le s$ and since, by the fact that $w = f(z)$ is univalent in R, the denominator of $g(y)$ in (11) never vanishes in $0 < |y| \le s$, $g(y)$ is analytic in $0 < |y| \le s$. Next we prove that $g(y)$ is also analytic at $y = 0$.

As $y \to 0$, by L'Hôpital's rule (see [2]) and by (4) we have

$$\frac{f(x+\omega y) - f(x+\omega^2 y)}{f(x+\omega^2 y) - f(x+y)} \to \frac{\omega f'(x) - \omega^2 f'(x)}{\omega^2 f'(x) - f'(x)} = \omega^2 \tag{13}$$

and

$$\frac{f(x+y) - f(x)}{f(x+\omega y) - f(x)} \to \frac{f'(x)}{\omega f'(x)} = \frac{1}{\omega}. \tag{14}$$

Hence, by (11), (13), (14), as $y \to 0$,

$$g(y) \to \omega^2 \cdot \frac{1}{\omega} = \omega. \tag{15}$$

If we define

$$g(0) = \omega, \tag{16}$$

by (15) and by Riemann's theorem on removable singularities, the function $g(y)$ is analytic at $y = 0$. Furthermore, by (16) the equality (12) still holds at $y = 0$.

Summarizing the above yields that $g(y)$ is analytic in $|y| \le s$ and that $|g(y)| = 1$ holds in $|y| \le s$. Therefore, by the maximum modulus principle [8, p. 141] for analytic functions we obtain

$$g(y) = K \tag{17}$$

in $|y| \le s$, where K is a complex constant with modulus 1.

Setting $y = 0$ in (17) and using (16) yields

$$K = \omega. \tag{18}$$

By (11), (17), (18) we obtain

$$\begin{aligned}\big(f(x+\omega y) - f(x+\omega^2 y)\big)\big(f(x+y) - f(x)\big) \\ - \omega\big(f(x+\omega^2 y) - f(x+y)\big)\big(f(x+\omega y) - f(x)\big) = 0\end{aligned} \tag{19}$$

for all y satisfying $|y| \le s$.

Using Leibnitz's rule for differentiation, differentiating both sides of (19) four times with respect to y, setting $y = 0$, and simplifying the resulting equality yields

$$f'''(x)f'(x) - \tfrac{3}{2}f''(x)^2 = 0. \tag{20}$$

Since $x \in R$ (the domain) was arbitrarily fixed, we can replace x by a variable z and thus, by (20), we have

$$f'''(z)f'(z) - \tfrac{3}{2}f''(z)^2 = 0$$

in R.

By the identity theorem (see[7, p. 106]) the above equality holds in $|z| < +\infty$. Hence

$$\frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = 0$$

holds for all z satisfying $f'(z) \neq 0$.

Thus, the Schwarzian derivative of f vanishes for all z satisfying $f'(z) \neq 0$. Therefore, by a well-known fact $f(z)$ is a Möbius transformation of z. Q.E.D.

Proof of Corollary. By hypothesis $w = f(z)$ satisfies Property A. Hence, by the result of Section 2, $w = f(z)$ satisfies Property B. Consequently, by the main theorem $w = f(z)$ is a Möbius transformation. Q.E.D.

ACKNOWLEDGMENT

We express out thanks to the referee for the helpful comments.

REFERENCES

1. J. Aczél and M. A. McKiernan, On the characterization of plane projective and complex Möbius transformation, *Math. Nachr.* **33** (1967), 315–337.
2. J. Aczél, Functional equations and L'Hôpital's rule in an exact Poisson derivation, *Amer. Math. Monthly* **97** (1990), 423–426.
3. H. S. M. Coxeter and S. L. Greitzer, "Geometry Revisited," Random House, New York, 1967.
4. H. Haruki, A proof of the principle of circle-transformation by use of a theorem on univalent functions, *Enseign. Math.* **18** (1972), 145–146.
5. H. Haruki and Th. M. Rassias, A new characterization of Möbius transformations by use of Apollonius quadrilaterals, *Topol. Anal.* to appear.
6. H. Eves, "A Survey of Geometry," Allyn & Bacon, Boston, 1972.
7. Z. Nehari, "Conformal Mapping," McGraw–Hill Book, New York, 1952.
8. R. Nevanlinna and V. Paatero, "Introduction to Complex Analysis," Addison–Wesley, New York, 1964.
9. L. L. Pennisi, L. I. Gordon, and S. Lasher, "Elements of Complex Variables," Holt, Rinehart & Winston, New York, 1963.